JOURNAL OF APPROXIMATION THEORY **28**, 329–332 (1980)

Some Results on Best Approximation in Locally Convex Spaces

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Communicated by G. Meinardus

Received August 20, 1978

Applications of fixed point theorems to approximation theory are well known. Results such as "a boundedly compact Chebyshev set is a sun," and "every boundedly compact Chebyshev set in a smooth Banach space is convex" have very elegant proofs using fixed point theorems. Brosowski [1] and Meinardus [3] used fixed point theory to prove some other interesting theorems in approximation theory.

The aim of this paper is to extend and unify the work of Meinardus and Brosowski. The results are given in locally convex spaces and the mapping considered is only continuous, not necessarily Lipschitz.

The following definitions and results will be needed.

Let C be a nonempty subset of a locally convex linear space X and let p be a continuous semi-norm on X. For $\mathbf{x}$ in X define

$$d_p(\mathbf{x}, C) = \inf\{p(\mathbf{x} - y)/y \text{ in } C\},$$

and

$$P_C(x) = \{x \in C/p(\mathbf{x} - x) = d_p(\mathbf{x}, C)\}.$$

C is said to be proximinal with respect to p if, for each $x \in X$, $P_C(x)$ is nonempty [4].

C is said to be star shaped if there is a point c in C such that, for all x in C and for all q, $0 \leqslant q < 1$, $(1 - q)c - qx$ is in C. The point c is called a star center.

A mapping $T: C \to C$ is said to be a p-contraction if there is a k_p, $0 \leqslant k_p < 1$, such that

$$p(Tx - Ty) \leqslant k_p p(x - y) \qquad \text{for all } x, y \text{ in } C \text{ and all } p \in P.$$

A mapping $T: C \to C$ is p-nonexpansive if, for all x, y in C,

$$p(Tx - Ty) \leqslant p(x - y), \qquad p \in P.$$

0021-9045/80/040329–04$02.00/0

The main theorem is the following.

THEOREM 1. *Let X be a locally convex linear Haudsorff space and let $T: X \to X$ be a continuous mapping. Let C be a T-invariant set and b a T-invariant point. If, for every p in P, the set D of best C-approximants to b with respect to p is nonempty, compact and star shaped, and T satisfies*

$$p(Tx - Tb) \leqslant p(x - b) \qquad \text{for all } x \text{ in } D,$$

and T is p-nonexpansive on D, then T has a fixed point which is a best approximation to b in C.

Proof. Since D is the set of best C-approximants to b, therefore, $T: D \to D$. In fact, if y is in D, then

$$p(Ty - b) = p(Ty - Tb) \leqslant p(y - b),$$

implying that Ty is in D.

Let c be a star center of D. Then $qc + (1 - q)x$ in D for all x in D. Let r_n be sequence of real numbers, $0 < r_n < 1$, such that $r_n \to 1$ as $n \to \infty$.

For each n in N, define

$$T_n: D \to D$$

such that

$$T_n(x) = r_n Tx + (1 - r_n)c \qquad \text{for all } x \text{ in } D.$$

Then

$$\begin{aligned} p(T_n x - T_n y) &= r_n p(Tx - Ty) \\ &\leqslant r_n p(x - y) \qquad \text{for all } x, y \text{ in } D. \end{aligned}$$

Since D is compact and each T_n is a p-contraction for every p in P, each T_n has a unique fixed point, say, x_n [2, 6]. Thus $T_n x_n = x_n$ for each n in N.

Since D is compact there exists a subnet $\{x_\alpha\}$ of $\{x_n\}$ such that x_α converges to z in D.

We claim that $z = Tz$.

$$x_\alpha = T_\alpha x_\alpha = (1 - k_\alpha)c + k_\alpha Tx_\alpha .$$

Since T is continuous, taking the limit we therefore get $z = Tz$.

We derive the following results as corollaries.

COROLLARY 1. *Let T be a p-nonexpansive, for every p in P, on a Hausdorff locally convex space X. Let C be a T-invariant subset of X and b a T-invariant* *point. If the set of best C-approximants to b is nonempty, compact and star shaped, then it contains a T-invariant point.*

COROLLARY 2. *Let T be a nonexpansive mapping of a normed linear space X, let C be a T-invariant subset of X and let b be a T-invariant point. If D, the set of best C-approximants to b, is nonempty, compact and star shaped, then T has a fixed point* **x** *in D and* **x** *is a best approximation to b in C* [5].

COROLLARY 3. *Let T be a nonexpansive mapping on a normed linear space X. Let C be a T-invariant subset of X and b a T-invariant point. If D, the set of best C-approximants to b, is nonempty, compact and convex, then it contains a T-invariant point* [1].

COROLLARY 4. *Let X be a normed linear space and $T: X \to X$ be a nonexpansive mapping. Let T have a fixed point, say, b, and a T-invariant finite dimensional subspace C of X. Then T has a fixed point which is a best approximation to b in C.*

It is known that the set D, of best C-approximants to b, is nonempty. Also, D is closed, bounded and convex. Since C is finite dimensional, D is compact and hence the result follows from Corollary 3.

The following well-known result of Meinardus [3] follows from Corollary 4.

Let $T: B \to B$ be continuous, where B is a compact metric space, if $C[B]$ is the space of all continuous real or complex functions on B with the sup norm. Let $A: C[B] \to C[B]$ be Lipschitz with Lipschitz constant 1. Suppose further that $Af(T(x)) = f(x)$, $Ah(T(x))$ in V whenever $h(x)$ is in V, where V is a finite dimensional subspace of $C[B]$.

Then there is a best approximation g of f with respect to V such that

$$Ag(T(x)) = g(x).$$

It is evident that the mapping $F: C[B] \to C[B]$ defined by

$$F(g(x)) = A(g(T(x)))$$

satisfies the conditions of Corollary 4.

A convex set is star shaped; however, a star shaped set need not be convex.

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